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The Wigner–Heisenberg algebra as an effective operator technique for simpler spectral resolution of general oscillator-related potentials and the connection with the SUSYQM algebra

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Abstract. We elucidate the utility of a super-realised Wigner–Heisenberg oscillator algebra with its inherent built-in direct generalisation of the usual oscillator ladder operators, but satisfying a generalised quantum commutation rule, as an effective algebraic tool for the easier spectral resolution of general oscillator-related potentials. To illustrate the procedure we consider here the full 3D isotropic harmonic oscillator problem and also the problems of the 1D-isotonic and d-dimensional radial oscillator systems. We also point out the intimate connection between the Wigner–Heisenberg algebra and the quantum mechanical SUSY algebra associated with these systems. The vastly simplified algebraic treatment within the framework of the Wigner–Heisenberg algebra of some other oscillator-related potentials like those of the non-relativistic and relativistic Coulomb problems for the electron or of certain generalised SUSY oscillator Hamiltonian models of the type of Celka and Hussin will be reported separately.

1. Introduction

In recent times the one-dimensional (1D) quantum mechanical (QM) supersymmetry (SUSY) algebra [1, 2] has been successfully utilised to achieve a SUSY generalisation [3-5] of the familiar harmonic oscillator raising and lowering operators for SUSY shape-invariant potentials. This has paved the way for the evolution of the SUSY method as a powerful algebraic technique for the spectral resolution of a variety of such potentials of physical interest [3, 4, 6-12]. While the SUSYQM algebra has thus received much operator applications for potential problems, another algebra, the general Wigner–Heisenberg (WH) oscillator algebra [13-19], which already possesses an inbuilt structure which generalises the usual oscillator ladder operators, has not, however, in our opinion, received its due attention in the literature as regards its potential for being developed as an effective operator technique for the spectral

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resolution of oscillator-related potentials. The purpose of the present paper is to bridge this gulf.

Basing our formalism specifically on a super-realised WH algebra wherein the defined ladder operators of the Wigner Hamiltonian system satisfy a generalised quantum commutation rule, we develop its utility as an effective operator technique for a simpler spectral resolution of general oscillator-related potentials. To illustrate the formalism we consider here simpler types of such potentials only, of the full 3D isotropic harmonic oscillator problem (for a particle of spin $\frac{1}{2}$) and of the 1D-isotonic and d-dimensional radial oscillator systems. The vastly simplified treatment within the framework of the WH algebra of some other oscillator-related potentials like those of the non-relativistic and relativistic Coulomb problems for the electron and of certain SUSY oscillator Hamiltonian models of the type of Celka and Hussin [20] which generalise the earlier potentials of Ui [21] and Balantekin [22], will be reported separately. One of us (JJ) will also demonstrate elsewhere how a significant percentage of other known SUSY shape-invariant potentials (see [4] for a list of such potentials), like the Pöschl–Teller I and II potentials, are actually amenable, by virtue of their hidden oscillator connections, to treatment using the WH algebra operator technique developed in this paper.

In section 2 we start by summarising the essential features of the abstract 1D WH algebra: the defining (anti-)commutation relations involving the Wigner Hamiltonian and its mutually adjoint linear ladder operators satisfying a concomitant general oscillator quantum rule of Wigner. Obtaining, then, its super-realisation, i.e. a realisation in terms of both bosonic and fermionic coordinates, we bring out the characterisation of a two-sector, bosonic with fermion number zero and fermionic with fermion number one, composition of the Wigner Hamiltonian. Though this composition is analogous to that existing for a SUSYQM Hamiltonian, the difference with the present case is that no energy degeneracy, as will be shown, can exist between the sector Hamiltonians of the Wigner system except in a unique limiting case when each sector Hamiltonian becomes identical to that of the usual oscillator. From the basis of our super-realised WH algebra, we develop in this section the main procedural steps for utilising the same as an operator technique for the complete spectral resolutions of the Wigner system as a whole and then of its sector Hamiltonians.

In section 3 we illustrate the application of our operator method to the case of the Hamiltonian of a 3D isotropic harmonic oscillator for spin $\frac{1}{2}$ embedded in the bosonic sector of a corresponding Wigner system, obtaining thereby its full spectral resolution with an ease comparable to that of the usual one-dimensional treatment.

We establish in section 4 the close connection existing between the 3D Wigner system considered in section 3 and a 3D SUSY oscillator system treated recently by Ui ([21, 23]; see also [24]).

In section 5 results analogous to those of section 3 and section 4 are quickly extracted for the 1D-isotonic and d-dimensional ($d \neq 1$) radial oscillator systems embedded in the bosonic sectors of the respective Wigner systems. The particular case of one dimension ($d = 1$) is shown to be obtained as an exceptional limit case of our general analysis as applied to the 1D Wigner isotonic case. As a by-product of our analysis in this section, we show also that the recent results of Dongpei [25] on his factorised form of the quadratic ladder operators of Camiz *et al* [26] for the 1D-isotonic oscillator Hamiltonian [27, 28] are simple transcriptions of the properties of the linear ladder operators of the associated Wigner isotonic system treated in this section.

Section 6 contains the concluding remarks.

2. The abstract WH algebra and its super-realisation

Almost four decades ago Wigner [13] posed an interesting question as to whether the equations of motion determine the quantum mechanical commutation relations and found as an answer a generalised quantum commutation rule for the one-dimensional harmonic oscillator. Starting with

$$\hat{H} = \tfrac{1}{2}(\hat{x}^2 + \hat{p}^2) = \tfrac{1}{2}[\hat{a}^-, \hat{a}^+]_+ = \tfrac{1}{2}(\hat{a}^-\hat{a}^+ + \hat{a}^+\hat{a}^-) \tag{1}$$

(we employ the convention of units such that $\hbar = m = \omega = 1$) where the abstract Wigner Hamiltonian $\hat{H}$ is expressed in the symmetrised bilinear form in the mutually adjoint abstract operators $\hat{a}^\pm$ defined by

$$\hat{a}^\pm = \frac{1}{\sqrt{2}}(\pm \mathrm{i}\hat{p} - \hat{x}) \qquad (\hat{a}^\pm)^\dagger = \hat{a}^\mp \tag{2}$$

Wigner [13] showed that the Heisenberg equations of motion

$$[\hat{H}, \hat{a}^\pm]_- = \pm\hat{a}^\pm \tag{3}$$

obtained by also combining the requirement that $\hat{x}$ satisfies the equation of motion of the classical form, i.e. $\ddot{\hat{x}} + \hat{x} = 0$, do not necessarily entail in the usual quantum rule $[\hat{a}^-, \hat{a}^+]_- = 1 \rightarrow [\hat{x}, \hat{p}]_- = \mathrm{i}$ but in a more general one. The form of this general quantum rule can be given by [14–18]

$$[\hat{a}^-, \hat{a}^+]_- = 1 + c\hat{R} \rightarrow [\hat{x}, \hat{p}]_- = \mathrm{i}(1 + c\hat{R}) \tag{4}$$

where c is a real constant that is related to the ground-state energy $E^{(0)}$ of $\hat{H}$ and $\hat{R}$ is an abstract operator, Hermitian and unitary,

$$\hat{R} = \hat{R}^\dagger = \hat{R}^{-1} \rightarrow \hat{R}^2 = 1 \tag{5}$$

also possessing the properties

$$[\hat{R}, \hat{a}^\pm]_+ = 0 \rightarrow [\hat{R}, \hat{H}]_- = 0. \tag{6}$$

It follows from (1) and (4) that

$$\hat{H} = \begin{cases} \hat{a}^+\hat{a}^- + \tfrac{1}{2}(1 + c\hat{R}) & (7a) \\ \hat{a}^-\hat{a}^+ - \tfrac{1}{2}(1 + c\hat{R}). & (7b) \end{cases}$$

Abstractly $\hat{R}$ is the Klein operator $\pm\exp[\mathrm{i}\pi(\hat{H} - E_0)]$ [15, 16] while in Schrödinger coordinate representation, first investigated by Yang [14], $\hat{R}$ is realised by $\pm P$ where P is the parity operator [14, 17–19].

The basic (anti-)commutation relations (1), (3) together with their derived relations (4)–(6) will be referred to here as constituting the WH algebra which is in fact a parabose algebra [18, 29, 30] for one degree of freedom.

To obtain a super-realisation of the WH algebra, we introduce, in addition to the usual bosonic coordinates $(x, -\mathrm{i}d/dx)$, the fermionic ones $b^\pm (= (b^\mp)^\dagger)$ that commute with the bosonic set and are represented in terms of the usual Pauli matrices Σ_i $(i = 1, 2, 3)$ by the combinations

$$b^- = \Sigma_+ = \tfrac{1}{2}(\Sigma_1 + \mathrm{i}\Sigma_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad (b^-)^2 = 0 \tag{8a}$$

$$b^+ = \Sigma_- = \tfrac{1}{2}(\Sigma_1 - \mathrm{i}\Sigma_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad (b^+)^2 = 0 \tag{8b}$$

$$[b^-, b^+]_+ = 1 \tag{8c}$$

so that the fermion number operator is given by

$$N_f = b^+b^- = \Sigma_-\Sigma_+ = \tfrac{1}{2}(1-\Sigma_3) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{9}$$

From the familiar bosonic coordinate commutation relations and the algebra of Pauli matrices, it follows straight away that the following super-realised operators:

$$\hat{a}^{\pm} \to a^{\mp}(c/2) = \frac{1}{\sqrt{2}}\left(\pm\Sigma_1 \frac{d}{dx} \mp \frac{c}{2x}\Sigma_1\Sigma_3 - \Sigma_1 x\right) \tag{10}$$

together with the Wigner Hamiltonian (1), i.e.

$$\hat{H} \to H(c/2) = \tfrac{1}{2}[a^-(c/2), a^+(c/2)]_+ \tag{11a}$$

$$= \frac{1}{2}\left[-\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}\left(\frac{c}{2}\Sigma_3\right)\left(\frac{c}{2}\Sigma_3 - 1\right)\right] \tag{11b}$$

$$= \begin{bmatrix} H_-(c/2-1) & 0 \\ 0 & H_+(c/2-1) = H_-(c/2) \end{bmatrix} \tag{11c}$$

$$H_-(c/2-1) = \frac{1}{2}\left[-\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}\left(\frac{c}{2}-1\right)\left(\frac{c}{2}\right)\right] \tag{11d}$$

do indeed satisfy the WH algebra ladder relations (3):

$$[H(c/2), a^{\pm}(c/2)]_- = \pm a^{\mp}(c/2). \tag{12}$$

Hence $a^+(c/2)$ and $a^-(c/2)$ are the raising and lowering operators for $H(c/2)$ of (11) and satisfy, as can be directly checked, the following generalised quantum commutation relation:

$$[a^-(c/2), a^+(c/2)]_- = 1 + c\Sigma_3. \tag{13}$$

In view of (2), equation (13) gets transcribed to

$$[\hat{x} \to \Sigma_1 x, \hat{p} \to -i\Sigma_1\, d/dx + (ic/2x)\Sigma_1\Sigma_3]_- = i(1 + c\Sigma_3). \tag{14}$$

Hence the abstract operator $\hat{R}$ of (4) is realised by Σ_3 so that, in view of (9),

$$\hat{R} \to \Sigma_3 = (1 - 2N_f) \tag{15}$$

which satisfies the equations (5), (6):

$$\Sigma_3^2 = 1 \qquad [\Sigma_3, a^{\pm}(c/2)]_+ = 0 \to [\Sigma_3, H(c/2)]_- = 0. \tag{16}$$

Since $H(c/2)$ and Σ_3 commute, one can choose simultaneous eigenstates of these two operators. This, in fact, has been presumed in writing the two-sector form (11c) for the Wigner Hamiltonian $H(c/2)$ given by (11b) wherein the sector Hamiltonians $H_-(c/2-1)$ and $H_+(c/2-1)$ belong to the subspaces of $H(c/2)$ characterised, respectively, by the eigenvalues 1 and -1 of Σ_3 or, equivalently, with fermion numbers 0 and 1, respectively, by virtue of (15). For this reason $H_-(c/2-1)$ and $H_+(c/2-1)$ will be designated as the bosonic (i.e. having fermion number 0) and the fermionic (i.e. having fermion number 1) sectors, respectively, of the Wigner Hamiltonian $H(c/2)$, borrowing the familiar SUSYQM nomenclature. One should be wary, however, that unlike with the SUSY case, there can exist no degeneracy, as will be established below, between the two-sector Hamiltonians here except in the unique limiting case of $c = 0$ (see section 5, equation (82)) for which case each sector Hamiltonian in (11c) becomes identical to that of the usual oscillator. For this special case the generalised commutation relation (13) (or (14)) sharpens to the usual canonical one.

That the sign of c in $H(c/2)$, equations (11a)-(11d), is purely conventional can be justified as follows. Firstly, in the defining expressions (10), (11b) and (13), the operator Σ_3 and the constant c appear only in the product form $c\Sigma_3$. Hence a simultaneous change of the signs of c and Σ_3 renders the Wigner Hamiltonian $H(c/2) = H(c/2, \Sigma_3)$ invariant, i.e.

$$H(c/2, \Sigma_3) = H(-c/2, -\Sigma_3). \tag{17a}$$

From (17a) and the equality $\Sigma_1\Sigma_3\Sigma_1 = -\Sigma_3$, it then follows that

$$\begin{aligned} H(-c/2) = H(-c/2, \Sigma_3) &= \Sigma_1 H(-c/2, -\Sigma_3)\Sigma_1 \\ &= \Sigma_1 H(c/2, \Sigma_3)\Sigma_1 = \Sigma_1 H(c/2)\Sigma_1. \end{aligned} \tag{17b}$$

The above equation establishes that the two theories related with just the change of sign of c are unitarily connected with each other. (See [18] for a discussion on this matter in the context of a purely coordinate description of the WH algebra.) In view of (17b) we shall assume hereinafter, without loss of generality, that c is positive, i.e.

$$c = |c| > 0. \tag{17c}$$

(The limiting case of $c = 0$ will be discussed separately in section 5.)

Now, to determine the ground state $\psi^{(0)}(c/2)$, a two-component entity, of $H(c/2)$,

$$H(c/2)\psi^{(0)}(c/2) = E^{(0)}(c/2)\psi^{(0)}(c/2) \qquad \psi^{(0)}(c/2) = \begin{bmatrix} \psi_{\mathrm{I}}^{(0)}(c/2) \\ \psi_{\mathrm{II}}^{(0)}(c/2) \end{bmatrix} \tag{18}$$

with $E^{(0)}(c/2)$ being the ground-state energy, we choose for $H(c/2)$ the convenient from (7a) which, in view of (10), (11) and (15), turns into

$$H(c/2) = a^{+}(c/2)a^{-}(c/2) + \tfrac{1}{2}(1 + c\Sigma_3). \tag{19}$$

From the positive semi-definite form (11a) of $H(c/2)$ and the energy step-up and step-down relations (12), it follows that $\psi^{(0)}(c/2)$ can be uniquely determined from (18) and (19) invoking the annihilation condition

$$a^{-}(c/2)\psi^{(0)}(c/2) = 0. \tag{20}$$

Expressing $a^{\mp}(c/2)$ given by (10) in the following respective factorised forms:

$$a^{+}(c/2) = \frac{1}{\sqrt{2}}\Sigma_1 x^{(1/2)c\Sigma_3} \exp(\tfrac{1}{2}x^2)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) \exp(-\tfrac{1}{2}x^2)x^{-(1/2)c\Sigma_3} \tag{21a}$$

$$a^{-}(c/2) = \frac{1}{\sqrt{2}}\Sigma_1 x^{(1/2)c\Sigma_3} \exp(-\tfrac{1}{2}x^2)\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right) \exp(\tfrac{1}{2}x^2)x^{-(1/2)c\Sigma_3} \tag{21b}$$

the use of (21b) in (20) leads to

$$[\exp(\tfrac{1}{2}x^2)x^{-(1/2)c\Sigma_3}]\begin{bmatrix} \psi_{\mathrm{I}}^{(0)}(c/2) \\ \psi_{\mathrm{II}}^{(0)}(c/2) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad c_1, c_2 \text{ constants.} \tag{22}$$

Equation (22) then results in

$$\begin{bmatrix} \psi_{\mathrm{I}}^{(0)}(c/2) \\ \psi_{\mathrm{II}}^{(0)}(c/2) \end{bmatrix} = \begin{bmatrix} c_1 x^{c/2}\exp(-\tfrac{1}{2}x^2) \\ c_2 x^{-c/2}\exp(-\tfrac{1}{2}x^2) \end{bmatrix}. \tag{23}$$

As c has been assumed positive, only $\psi_{\mathrm{I}}^{(0)}(c/2)$ of (23) meets the physical requirement of vanishing at the origin and $\psi_{\mathrm{II}}^{(0)}(c/2)$, which does not stand this test, is discarded by setting $c_2=0$ in (23). Hence, the normalisable ground-state wavefunction is given, up to a normalisation constant, by

$$\psi^{(0)}(c/2)=\begin{bmatrix}\psi_{\rm I}^{(0)}(c/2)\\ 0\end{bmatrix}\propto\begin{bmatrix}x^{c/2}\exp(-\frac{1}{2}x^2)\\ 0\end{bmatrix} \tag{24}$$

which has even Σ_3-parity, i.e.

$$\Sigma_3\psi^{(0)}(c/2)=\psi^{(0)}(c/2). \tag{25}$$

From (18)-(20), (24) and (25), it follows that the ground-state energy is given by

$$E^{(0)}(c/2)=\tfrac{1}{2}(1+c)>\tfrac{1}{2} \qquad c>0. \tag{26}$$

At this stage an independent verification of the existence or not of a zero ground-state energy for $H(c/2)$ suggested by its positive semi-definite form may be in order. Such a state has to be annihilated both by $a^-(c/2)$ and $a^+(c/2)$ and the latter condition leads to unnormalisable $\psi_{\rm I}^{(0)}$ and $\psi_{\rm II}^{(0)}$ in view of (21a), thus causing the state in question to vanish identically. However, there exists a normalisable state of lowest energy $E^{(0)}(c/2)=1$, which is annihilated by $a^-(c/2)$ but not by the energy step-up operator $a^+(c/2)$. A little further analysis reveals that the theory then becomes identical to the description with $c=1$ (or equivalently related to $c=-1$ in view of (17b)). For a similar conclusion from matrix representations of the parabose algebra, see Sharma *et al* [30].

Now, from the role of $a^+(c/2)$ as the energy step-up operator (the upper sign choice in equation (12)) the complete energy spectrum of $H(c/2)$ is given by

$$E^{(n)}(c/2)=E^{(0)}+n=\tfrac{1}{2}(1+c)+n \qquad n=0,1,2,\ldots. \tag{27}$$

The excited-state energy eigenfunctions $\psi^{(n)}(c/2)$ are readily given by the step-up operation with $a^+(c/2)$ starting with $\psi^{(0)}(c/2)$:

$$\begin{aligned}\psi^{(n)}(c/2)&\propto[a^+(c/2)]^n\psi^{(0)}(c/2)=[a^+(c/2)]^n\begin{bmatrix}\psi_{\rm I}^{(0)}(c/2)\\ 0\end{bmatrix}\\ \psi^{(n)}(c/2)&=\begin{bmatrix}\psi_{\rm I}^{(n)}(c/2)\\ \psi_{\rm II}^{(n)}(c/2)\end{bmatrix}.\end{aligned} \tag{28}$$

From the equalities

$$\Sigma_1\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \qquad \Sigma_1\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \tag{29}$$

and the fact that Σ_1 can be factored out in the expression for $a^+(c/2)$ obtained by choosing the upper signs in (10), it follows that the excited-state wavefunctions given by (28) have even (odd) Σ_3-parity for even (odd) quanta $n=2m(n=2m+1)$, $m=0,1,2,\ldots$, i.e.

$$\psi^{(n=2m)}(c/2)=\begin{bmatrix}\psi_{\rm I}^{(n=2m)}(c/2)\\ 0\end{bmatrix} \tag{30a}$$

$$\psi^{(n=2m+1)}(c/2)=\begin{bmatrix}0\\ \psi_{\rm II}^{(n=2m+1)}(c/2)\end{bmatrix} \qquad m=0,1,2,\ldots. \tag{30b}$$

The component structures of the excited-state eigenfunctions of $H(c/2)$ given by (30) together with the two diagonal block structure of $H(c/2)$ given by (11c) and the energy spectrum (27) for $H(c/2)$ lead to the immediate obtaining of the mth excited-state eigenfunctions $\psi_-^{(m)}(c/2-1)$ and $\psi_+^{(m)}(c/2-1)$, respectively, of the bosonic and fermionic sector Hamiltonians $H_-(c/2-1)$ and $H_+(c/2-1)$ and of the complete energy spectrum for these respective Hamiltonians, as the ones given by

$$\psi_-^{(m)}(c/2-1)=\psi_{\mathrm{I}}^{(2m)}(c/2) \tag{31a}$$

$$\begin{bmatrix}\psi_{\mathrm{I}}^{(2m)}(c/2)\\0\end{bmatrix}\propto(a^+(c/2))^{2m}\psi^{(0)}(c/2)=(a^+(c/2))^{2m}\begin{bmatrix}\psi_{\mathrm{I}}^{(0)}(c/2)\\0\end{bmatrix} \tag{31b}$$

$$\begin{aligned}E_-^{(m)}&(c/2-1)\\&=E^{(2m)}(c/2)\\&=E^{(0)}(c/2)+2m=\tfrac{1}{2}(1+c)+2m\qquad m=0,1,2,\ldots\end{aligned} \tag{31c}$$

$$\psi_+^{(m)}(c/2-1)=\psi_{\mathrm{II}}^{(2m+1)}(c/2) \tag{32a}$$

$$\begin{bmatrix}0\\\psi_{\mathrm{II}}^{(2m+1)}(c/2)\end{bmatrix}\propto(a^+(c/2))^{2m+1}\psi^{(0)}(c/2)=(a^+(c/2))^{2m+1}\begin{bmatrix}\psi_{\mathrm{I}}^{(0)}(c/2)\\0\end{bmatrix} \tag{32b}$$

$$\begin{aligned}E_+^{(m)}&(c/2-1)\\&=E^{(2m+1)}(c/2)=E^{(0)}(c/2)+2m+1\\&=\tfrac{1}{2}(1+c)+2m+1\qquad m=0,1,2,\ldots.\end{aligned} \tag{32c}$$

We now demonstrate how equations (31*a*), (31*b*) and (32*a*), (32*b*) do in fact lead to the eigensolutions of $H_-(c/2-1)$ and $H_+(c/2-1)$, respectively, in terms of the generalised Laguerre polynomials defined ([17, 31]) by

$$L_n^{(\alpha)}(\rho)=\frac{1}{n!}\rho^{-\alpha}\left(\exp(\rho)\frac{\mathrm{d}^n}{\mathrm{d}\rho^n}\exp(-\rho)\right)\rho^{n+\alpha}\qquad\rho=x^2\qquad 0<\rho<\infty \tag{33}$$

with $L_n^{(0)}(\rho)=L_n(\rho)$ being the ordinary Laguerre polynomials.

Considering first the case of even quanta ($n=2m$), equations (31*a*), (31*b*) together with (24) and the factorised expression (21*a*) for $a^+(c/2)$ lead to

$$\begin{aligned}\psi_-^{(m)}&(c/2-1)\begin{bmatrix}1\\0\end{bmatrix}\\&=\psi_{\mathrm{I}}^{(2m)}(c/2)\begin{bmatrix}1\\0\end{bmatrix}\propto[(a^+(c/2))^2]^m x^{c/2}\exp(-\tfrac{1}{2}x^2)\begin{bmatrix}1\\0\end{bmatrix}\\&=\left[\left(\frac{1}{\sqrt{2}}\Sigma_1 x^{(c/2)\Sigma_3}\exp(\tfrac{1}{2}x^2)\frac{\mathrm{d}}{\mathrm{d}x}\exp(-\tfrac{1}{2}x^2)x^{-(c/2)\Sigma_3}\right)^2\right]^m\\&\quad\times x^{c/2}\exp(-\tfrac{1}{2}x^2)\begin{bmatrix}1\\0\end{bmatrix}.\end{aligned} \tag{34}$$

After eliminating the fermion spinorial part in (34) one obtains

$$\begin{aligned}\psi_-^{(m)}&(c/2-1)\\&=\psi_{\mathrm{I}}^{(2m)}(c/2)\propto x^{c/2}\exp(-\tfrac{1}{2}x^2)\\&\quad\times\left[\left(\exp(x^2)(x^2)^{-(c/2-1/2)}\frac{\mathrm{d}}{\mathrm{d}x^2}(x^2)^{(c/2+1/2)}\frac{\mathrm{d}}{\mathrm{d}x^2}\exp(-x^2)\right)^m\right]\end{aligned} \tag{35}$$

where use has been made of the identity $\mathrm{d}/\mathrm{d}x=2x\,\mathrm{d}/\mathrm{d}x^2$.

For $m=0$ (1) it follows trivially that the square bracketed expression in (35) equals $L_0^{(c/2-1/2)}(x^2)=1$ ($L_1^{(c/2-1/2)}(x^2)$, apart from a proportionality factor) in view of (33). The demonstration that for arbitrary integer m the square bracketed expression in (35) is indeed proportional to $L_m^{(c/2-1/2)}(x^2)$ defined as in (33) proceeds through the familiar induction method which requires, as can be directly checked, the proving of the following equality:

$$\left[(x^2)^{(c/2+1/2)}\frac{\mathrm{d}}{\mathrm{d}x^2}(x^2)^{-(c/2-1/2)}\left(\frac{\mathrm{d}}{\mathrm{d}x^2}\right)^m(x^2)^{(c/2-1/2)}\exp(-x^2)\right](x^2)^m$$
$$=-\left[\left(\frac{\mathrm{d}}{\mathrm{d}x^2}\right)^m(x^2)^{(c/2-1/2)}\exp(-x^2)\right](x^2)^{m+1}. \tag{36}$$

This equality can in fact be established in a straightforward though tedious manner, the details of which we omit here. Hence equation (35) furnishes

$$\psi_-^{(m)}(c/2-1)=\psi_{\mathrm{I}}^{(2m)}(c/2)\propto x^{c/2}\exp(-\tfrac{1}{2}x^2)L_m^{(c/2-1/2)}(x^2). \tag{37}$$

Considering now the case of odd quanta ($n=2m+1$), defined in (32*a*), (32*b*), a repetition of the above procedure results in

$$\psi_+^{(m)}(c/2-1)$$
$$=\psi_{\mathrm{II}}^{(2m+1)}(c/2)\propto x^{(c/2+1)}\exp(-\tfrac{1}{2}x^2)$$
$$\times\left[\exp(x^2)\frac{\mathrm{d}}{\mathrm{d}x^2}(x^2)^{-(c/2-1/2)}\frac{\mathrm{d}}{\mathrm{d}x^2}(x^2)^{(c/2+1/2)}\exp(-x^2)\right]^m \tag{38a}$$
$$=x^{(c/2+1)}\exp(-\tfrac{1}{2}x^2)$$
$$\times\left[\exp(x^2)(x^2)^{-(c/2+1/2)}\frac{\mathrm{d}}{\mathrm{d}x^2}(x^2)^{(c/2+3/2)}\frac{\mathrm{d}}{\mathrm{d}x^2}\exp(-x^2)\right]^m \tag{38b}$$
$$=x^{(c/2+1)}\exp(-\tfrac{1}{2}x^2)L_m^{(c/2+1/2)}(x^2) \tag{38c}$$

where in passing from (38*a*) to (38*b*) use has been made of the identity

$$\frac{\mathrm{d}}{\mathrm{d}x^2}(x^2)^{-(c/2-1/2)}\frac{\mathrm{d}}{\mathrm{d}x^2}(x^2)^{(c/2+1/2)}=(x^2)^{-(c/2+1/2)}\frac{\mathrm{d}}{\mathrm{d}x^2}(x^2)^{(c/2+3/2)}\frac{\mathrm{d}}{\mathrm{d}x^2}. \tag{39}$$

The above analysis constitutes our affirmed procedural development of a super-realised WH algebra as a potentially effective operator technique for applications for obtaining spectral resolution of oscillator-related potentials some of which will be discussed in the ensuing sections.

Before proceeding further, an explanation may be in order as regards the role of the parity operator P, $P\psi(x)=\psi(-x)$ in the theory presented here. Though P does commute with $H(c/2)$, $H_-(c/2-1)$ and $H_+(c/2-1)$, the peculiarity that their respective eigenfunctions $\psi^{(n)}(c/2)$, $\psi_-^{(m)}(c/2-1)$ and $\psi_+^{(m)}(c/2-1)$ ((30)-(32), (37) and (38*c*)) are not automatically eigenfunctions of P for arbitrary $c>0$ even in this one-dimensional situation is due to the singular nature of the centrifugal terms in these Hamiltonians, which forbid transitions between the sectors $x>0$ and $x<0$. This makes possible, for any $c>0$, even or odd extensions to $x<0$ of the solutions for $x>0$ without any further condition of continuity at the singular origin that has just a vanishing wavefunction. Hence the regions $x>0$ and $x<0$ behave independent of each other for any $c>0$. The energy spectra given by (27), (31*c*) and (32*c*) are obviously unaffected by any parity prescription.

3. The 3D isotropic spin-$\frac{1}{2}$ oscillator Hamiltonian in the bosonic sector of a Wigner system

The super-realisation of the ladder operators (10) and the consequent form of the Wigner Hamiltonian of (11) wherein the potential terms, $V_-(c/2-1)$ and $V_+(c/2-1)$, of the sector Hamiltonians differ only by a change of the parameter, i.e. $V_+(c/2-1) = V_-(c/2)$, guide the construction as well of a 3D Wigner Hamiltonian $H(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$ with its bosonic sector taken as the Hamiltonian

$$H_-(\boldsymbol{\sigma}\cdot\boldsymbol{L}) = \tfrac{1}{2}(\boldsymbol{p}^2+r^2) = \frac{1}{2}\left(-\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}(\boldsymbol{\sigma}\cdot\boldsymbol{L})(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) + r^2\right) \qquad (0<r<\infty) \tag{40}$$

for a non-relativistic 3D isotropic oscillator with spin-$\frac{1}{2}$ represented here by $\frac{1}{2}\boldsymbol{\sigma}$. With the use of the following familiar spin-$\frac{1}{2}$ equalities [32, 33]:

$$\begin{aligned} &\boldsymbol{\sigma}\cdot\boldsymbol{p} = \sigma_r p_r + \frac{\mathrm{i}}{r}\sigma_r(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) \qquad p_r = -\mathrm{i}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) = \frac{1}{r}\left(-\mathrm{i}\frac{\partial}{\partial r}\right)r = p_r^{\dagger} \\ &\sigma_r = \frac{1}{r}\boldsymbol{\sigma}\cdot\boldsymbol{r} \qquad \sigma_r^2 = 1 \qquad [\sigma_r, \boldsymbol{\sigma}\cdot\boldsymbol{L}+1]_+ = 0 \qquad L^2 = \boldsymbol{\sigma}\cdot\boldsymbol{L}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) \end{aligned} \tag{41}$$

(use of which has been made in writing the form (40) for $H_-(\boldsymbol{\sigma}\cdot\boldsymbol{L})$) and invoking analogy with (11) and (10), one simply makes the replacements in these of

$$x \to r \qquad \frac{\mathrm{d}}{\mathrm{d}x} \to \frac{\partial}{\partial r} + \frac{1}{r} = \frac{1}{r}\frac{\partial}{\partial r}r \qquad \left(\frac{\partial}{\partial r}+\frac{1}{r}\right)^{\dagger} = -\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) \tag{42}$$

along with the constant $c/2$ substituted in its place by the operator $(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$ that commutes with every quantity occurring in (40), to obtain the 3D fermionic sector Hamiltonian

$$\begin{aligned} H_+(\boldsymbol{\sigma}\cdot\boldsymbol{L}) &= H_-(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) \\ &= \frac{1}{2}\left(-\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)(\boldsymbol{\sigma}\cdot\boldsymbol{L}+2) + r^2\right) \end{aligned} \tag{43}$$

so that the 3D Wigner Hamiltonian is then defined by

$$\begin{aligned} H(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) &= \begin{bmatrix} H_-(\boldsymbol{\sigma}\cdot\boldsymbol{L}) & 0 \\ 0 & H_+(\boldsymbol{\sigma}\cdot\boldsymbol{L}) = H_-(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) \end{bmatrix} \\ &= \tfrac{1}{2}[a^-(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1), a^+(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)]_+. \end{aligned} \tag{44}$$

In (44) the operators $a^{\mp}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$ defined by

$$\begin{aligned} a^{\mp}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) &= \frac{1}{\sqrt{2}}\left[\pm\Sigma_1\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) \mp \frac{1}{r}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)\Sigma_1\Sigma_3 - \Sigma_1 r\right] \\ &= [a^{\pm}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)]^{\dagger} \end{aligned} \tag{45}$$

are in fact the ladder operators for $H(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$ of (44):

$$[H(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1), a^{\pm}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)]_- = \pm a^{\mp}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) \tag{46}$$

and satisfy the commutation relation

$$[a^+(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1), a^-(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)]_- = 1 + 2(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)\Sigma_3. \tag{47a}$$

Note that

$$[\Sigma_3, a^{\mp}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)]_+ = 0 \rightarrow [\Sigma_3, H(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)]_- = 0. \tag{47b}$$

Equations (44)-(47) in fact define a 3D WH algebra. Since the operator $(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$ commutes with all the basic elements $H(\boldsymbol{\sigma}\cdot\boldsymbol{L})$ and $a^{\mp}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$ of this algebra, it can be simply replaced by its eigenvalues $\pm(l+1)$, $l=0,1,2,\ldots$, while acting on its respective eigenspaces defined by the familiar spin spherical harmonics $y_\pm(\theta,\phi)$:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)y_\pm(\theta,\phi) = \pm(l+1)y_\pm(\theta,\phi) \tag{48a}$$

$$y_+(\theta,\phi) = y_{l\frac{1}{2};j=l+\frac{1}{2},m_j}(\theta,\phi) \qquad y_-(\theta,\phi) = y_{l+1\frac{1}{2};j=(l+1)-\frac{1}{2},m_j}(\theta,\phi). \tag{48b}$$

On these subspaces the 3D WH algebra of (44)-(47) is reduced to the corresponding 1D form of equations (10)-(13) with $(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)\rightarrow\pm(l+1)=c/2$.

Considering first the case of $c/2=l+1\geqslant 1$, one proceeds through the algebraic technique developed through equations (18)-(39) and simply translates the contents of all these equations to the present case with the only modifications given by (42) so as to obtain the complete resolution of the energy spectrum and the corresponding (radial) eigenfunctions, indicated below in the now familiar notation, of the 1D Wigner Hamiltonian $H(l+1)$ and its sector Hamiltonians $H_-(l)$ and $H_+(l)$:

$$E^{(n)}(l+1) = E^{(0)}(l+1)+n = (l+\tfrac{3}{2})+n \qquad n=0,1,2,\ldots \tag{49}$$

$$R^{(n=2m)}(l+1) = \begin{bmatrix} R_{\rm I}^{(n=2m)}(l+1) \\ 0 \end{bmatrix} \qquad R^{(n=2m+1)}(l+1) = \begin{bmatrix} 0 \\ R_{\rm II}^{(n=2m+1)}(l+1) \end{bmatrix} \tag{50}$$

$$R_{\rm I}^{(n=2m)}(l+1) = R_-^{(m)}(l) = R_{Nl}(r) \propto r^l \exp(-\tfrac{1}{2}r^2) L^{(l+1/2)}_{m=(1/2)(N-l)}(r^2) \tag{51a}$$

$$E_-^{(m)}(l) = E^{(2m)}(l+1) = (l+\tfrac{3}{2})+2m \qquad m=0,1,2,\ldots \tag{51b}$$

$$E_{Nl} = E_-^{(m)}(l) = \tfrac{3}{2}+N \qquad N=l,l+2,\ldots \tag{51c}$$

$$R_{\rm II}^{(n=2m+1)}(l+1) = R_+^{(m)}(l) = R_{N\,l+1}(r) \propto r^{l+1}\exp(-\tfrac{1}{2}r^2) L^{(l+3/2)}_{m=(1/2)(N-l-1)}(r^2) \tag{52a}$$

$$E_+^{(m)}(l) = E^{(2m+1)}(l+1) = (l+\tfrac{3}{2})+2m+1 \qquad m=0,1,2,\ldots \tag{52b}$$

$$E_{N\,l+1} = E_+^{(m)}(l) = \tfrac{3}{2}+N \qquad N=l+1,l+3,\ldots \tag{52c}$$

where N is the principal quantum number.

For the case $c/2=-(l+1)$, the 1D Wigner Hamiltonian $H[-(l+1)]$ assumes the form

$$\begin{aligned} H[-(l+1)] &= \begin{bmatrix} H_-(-l-2) & 0 \\ 0 & H_+(-l-2)=H_-(-l-1) \end{bmatrix} \\ &= \begin{bmatrix} H_-(l+1) & 0 \\ 0 & H_-(l) \end{bmatrix} \\ &= \Sigma_1 \begin{bmatrix} H_-(l) & 0 \\ 0 & H_+(l)=H_-(l+1) \end{bmatrix} \Sigma_1 = \Sigma_1 H(l+1)\Sigma_1 \end{aligned} \tag{53}$$

(see also equation (17b)). While the eigenvalues of $H[-(l+1)]$ are thus the same as those of $H(l+1)$ given by (49), the effect of the unitary transformation (53) with Σ_1 on the eigenfunctions (50) of $H(l+1)$ is just to flip, by virtue of (29), the Σ_3-parities of the corresponding even and odd quanta eigenfunctions of $H[-(l+1)]$. Note that the ground state of $H[-(l+1)]$ has odd Σ_3-parity. Taking this fact into account, the complete set of energy eigenfunctions of $H_-(\boldsymbol{\sigma}\cdot\boldsymbol{L})$ of (40), which are also simultaneous eigenfunctions of $(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$, are readily regained by affixing the spin-spherical harmonics $y_\pm(\theta,\phi)$ of (48) suitably to the radial solutions obtained here, which turn out to be the known solutions in the standard forms $R_{Nl}(r)y_+(\theta,\phi)$ and $R_{N\,l+1}(r)y_-(\theta,\phi)$.

The above analysis completes our assertion as to how the WH algebra operator technique developed in section 2 can be effectively utilised for the complete spectral resolution of the spectral problem for a 3D isotropic harmonic oscillator with spin $\frac{1}{2}$.

4. The Wigner and SUSY systems associated with a 3D isotropic oscillator with spin $\frac{1}{2}$

In this section we demonstrate the connection between the 3D Wigner Hamiltonian $H(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$ of equation (44) and a 3D SUSY isotropic harmonic oscillator for spin $\frac{1}{2}$ recently discussed by Ui [21] with its Hamiltonian H_U given by

$$H_U=\begin{bmatrix}\frac{1}{2}(\boldsymbol{p}^2+r^2)-(\boldsymbol{\sigma}\cdot\boldsymbol{L}+\frac{3}{2}) & 0\\ 0 & \frac{1}{2}(\boldsymbol{p}^2+r^2)+(\boldsymbol{\sigma}\cdot\boldsymbol{L}+\frac{3}{2})\end{bmatrix} \tag{54}$$

(wherein, for convenience, we have reversed the bosonic and fermionic sectors as considered by Ui and have set his parameter ω to unity).

Transforming H_U by the unitary operator

$$U=\begin{bmatrix}1 & 0\\ 0 & \sigma_r\end{bmatrix} \qquad U^\dagger=U^{-1}=U \tag{55}$$

we obtain, in view of (40)-(44), the transformed Hamiltonian H_{SS} as given by

$$H_{SS}=UH_UU^\dagger=H(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)-\tfrac{1}{2}\Sigma_3[1+2(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)\Sigma_3]. \tag{56a}$$

Making use in the above of the symmetrised bilinear form (44) for $H(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$ in terms of the Wigner system ladder operators $a^\mp(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$ of equation (45) and identifying the square bracketed expression on the RHS of (56a) with the commutator (47a) of these ladder operators, H_{SS} of (56a) can be recast into the form

$$H_{SS}=\tfrac{1}{2}[a^-(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1),a^+(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)]_+-\tfrac{1}{2}\Sigma_3[a^-(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1),a^+(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)]_- \tag{56b}$$

$$=[Q_-,Q_+]_+. \tag{56c}$$

With Σ_3 anti-commuting with $a^\mp(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$, defined in (47$b$), the mutually adjoint charge operators $Q_\mp$ of (56c) obtain from (56b) the following respective expressions in terms of the Wigner system ladder operators:

$$Q_-=\tfrac{1}{2}(1-\Sigma_3)a^-(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) \qquad Q_+=Q_-^\dagger=\tfrac{1}{2}(1+\Sigma_3)a^+(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) \tag{57a}$$

$$(Q_-)^2=(Q_+)^2=0. \tag{57b}$$

The anti-commutation relations (56c) and (57b) define the SUSYQM algebra leading to the supersymmetry of H_{SS}, i.e.

$$[Q_\mp,H_{SS}]_-=0. \tag{58}$$

Equations (54)-(58) provide the intimate connection existing between the 3D SUSY isotropic harmonic oscillator of spin $\frac{1}{2}$ (i.e. equation (54)) of Ui [21] and the corresponding Wigner system Hamiltonian $H(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$ discussed in the last section.

In fact (57*a*) and (56) can be written in the following suggestive forms usually employed in SUSYQM discussions:

$$Q_- = \Sigma_- A^-(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) = \begin{bmatrix} 0 & 0 \\ A^-(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) & 0 \end{bmatrix} \tag{59a}$$

$$Q_+ = (Q_-)^\dagger = \Sigma_+ A^+(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) = \begin{bmatrix} 0 & A^+(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) \\ 0 & 0 \end{bmatrix} \tag{59b}$$

$$H_{\mathrm{SS}} = \begin{bmatrix} A^+(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)A^-(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) & 0 \\ 0 & A^-(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)A^+(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) \end{bmatrix} \tag{59c}$$

where $\Sigma_\mp$ are as defined in (8) and

$$A^\pm(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) = \frac{1}{\sqrt{2}}\left[\pm\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{1}{r}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)-r\right]. \tag{59d}$$

That SUSY is unbroken in this model for subspaces of y_+ follows from the existence of normalisable and non-singular (at the origin) zero-energy solutions pertaining to this case only, i.e. $\boldsymbol{\sigma}\cdot\boldsymbol{L}+1 \to l+1$

$$\phi_0 = \begin{bmatrix} \phi_{\mathrm{B}} \\ \phi_{\mathrm{F}} \end{bmatrix} \tag{60a}$$

that are annihilated by both Q_+ and Q_-. These annihilation conditions respectively lead by virtue of (59*a*), (59*b*) and (59*d*) to

$$Q_+\phi_0 = 0 \to \phi_{\mathrm{B}} \text{ arbitrary, } \phi_{\mathrm{F}} = 0 \tag{60b}$$

$$Q_-\phi_0 = 0 \to \phi_{\mathrm{B}} \propto r^l \exp(-\tfrac{1}{2}r^2)y_+,\ \phi_{\mathrm{F}} \text{ arbitrary} \tag{60c}$$

and hence together to

$$\phi_0 = \begin{bmatrix} \phi_{\mathrm{B}} \\ \phi_{\mathrm{F}} \end{bmatrix} \propto \begin{bmatrix} r^l \exp(-\tfrac{1}{2}r^2)y_+ \\ 0 \end{bmatrix} \tag{60d}$$

consistent with the conclusions of Ui [21] for his model based on (54).

5. The 1D-isotonic and *d*-dimensional radial oscillator systems

The 1D-isotonic oscillator is by definition the usual oscillator with a centripetal barrier [27, 28] and its Hamiltonian is just given by that of $H_-(c/2-1)$ defined in (11*d*). The Wigner isotonic system incorporating $H_-(c/2-1)$ in its bosonic sector is given by (11*c*). Assuming $c>0$ without loss of generality as justified by (17*b*), the operator technique of section 2 using the WH algebra is trivially applied to the present case to obtain the eigenfunctions $\psi_-^{(m)}(c/2-1)$ as given by (37) and the energy spectrum as given by (31*c*). The same observations as regards the role of parity as discussed in section 2 apply here as well.

Though quadratic ladder operators [26, 28] and their factorised expressions [25] for the 1D-isotonic oscillator have been discussed in the literature, the fact that they may be derived from their connection with the linear ladder operators of the associated Wigner system, however, has, to our knowledge, not yet been pointed out. We shall now demonstrate how the linear ladder operators (10) satisfying the Wigner ladder relations (12) lead to the obtaining of the quadratic ladder operators for the sector

Hamiltonians $H_-(c/2-1)$ and $H_+(c/2-1)=H_-(c/2)$, that raise the respective energy quanta of these Hamiltonians by two units.

Setting $c/2=l+1(>0)$ for convenience of later comparison with Dongpei's [25] results, we have from (12) the Wigner ladder relations

$$[H(l+1), a^\pm(l+1)]_- = \pm a^\pm(l+1) \tag{61a}$$

where

$$H(l+1)=\begin{bmatrix} H_-(l) & 0 \\ 0 & H_+(l)=H_-(l+1)\end{bmatrix}$$
$$=\begin{bmatrix} \frac{1}{2}\left(-\frac{d^2}{dx^2}+x^2+\frac{l(l+1)}{x^2}\right) & 0 \\ 0 & \frac{1}{2}\left(-\frac{d^2}{dx^2}+x^2+\frac{(l+1)(l+2)}{x^2}\right)\end{bmatrix} \tag{61b}$$

and

$$a^\pm(l+1)=\frac{1}{\sqrt{2}}\left(\pm\Sigma_1\frac{d}{dx}\mp\frac{(l+1)}{x}\Sigma_1\Sigma_3-\Sigma_1 x\right). \tag{61c}$$

From (61a) it follows that

$$[H(l+1), (a^\pm(l+1))^2]_- = \pm 2(a^\pm(l+1))^2. \tag{62}$$

Multiplying both sides of (62) from left by the projection operators $\frac{1}{2}(1\pm\Sigma_3)$, we obtain

$$\tfrac{1}{2}(1+\Sigma_3)[H(l+1), (a^\pm(l+1))^2]_- = \pm 2\tfrac{1}{2}(1+\Sigma_3)(a^\pm(l+1))^2 \tag{63}$$

$$\tfrac{1}{2}(1-\Sigma_3)[H(l+1), (a^\pm(l+1))^2]_- = \pm 2\tfrac{1}{2}(1-\Sigma_3)(a^\pm(l+1))^2. \tag{64}$$

Noting the commutativity of $\frac{1}{2}(1\pm\Sigma_3)$ with both $H(l+1)$ and $(a^\pm(l+1))^2$ and making use of the identities

$$\tfrac{1}{2}(1+\Sigma_3)H(l+1)=\tfrac{1}{2}(1+\Sigma_3)H_-(l) \tag{65a}$$

$$\tfrac{1}{2}(1-\Sigma_3)H(l+1)=\tfrac{1}{2}(1-\Sigma_3)H_+(l)=\tfrac{1}{2}(1-\Sigma_3)H_-(l+1) \tag{65b}$$

that follow from (61b), we obtain after elimination of the spinorial parts in (63) and (64) the following relations:

$$[H_-(l), A^+(l+1)A^+(-(l+1))]_- = 2A^+(l+1)A^+(-(l+1)) \tag{66a}$$

$$[H_-(l), A^-(-(l+1))A^-(l+1)]_- = -2A^-(-(l+1))A^-(l+1) \tag{66b}$$

$$[H_-(l+1), A^+(-(l+1))A^+(l+1)]_- = 2A^+(-(l+1))A^+(l+1) \tag{67a}$$

$$[H_-(l+1), A^-(l+1)A^-(-(l+1))]_- = -2A^-(l+1)A^-(-(l+1)). \tag{67b}$$

In the above the operators $A^+(\pm(l+1))$ and $A^-(\pm(l+1))$ are, respectively, defined by

$$A^+(\pm(l+1))=\frac{1}{\sqrt{2}}\left(\frac{d}{dx}\pm\frac{1}{x}(l+1)-x\right)=[A^-(\pm(l+1))]^\dagger \tag{68a}$$

$$A^-(\pm(l+1))=\frac{1}{\sqrt{2}}\left(-\frac{d}{dx}\pm\frac{1}{x}(l+1)-x\right)=[A^+(\pm(l+1))]^\dagger. \tag{68b}$$

Changing $l+1 \to l$ in (67), one also obtains

$$[H_-(l), A^+(-l)A^+(l)]_- = 2A^+(-l)A^+(l) \tag{69a}$$

$$[H_-(l), A^-(l)A^-(-l)]_- = -2A^-(l)A^-(-l). \tag{69b}$$

From (66*a*) and (69*a*) it is readily identified that the quadratic operators $A^+(l+1)A^+(-(l+1))$ and $A^+(-l)A^+(l)$ are both step-up operators by two units for the energy spectrum of $H_-(l)$. These quadratic operators are indeed identical to each other, as can be directly checked in view of (68*a*):

$$A^+(-l)A^+(l) = A^+(l+1)A^+(-(l+1)). \tag{70}$$

Observing that (66*b*), (67*b*) and (69*b*) are just the Hermitian adjoint equations respectively to (66*a*), (67*a*) and (69*a*), it follows that the quadratic step-down operators

$$A^-(l)A^-(-l) = A^-(-(l+1))A^-(l+1) \tag{71}$$

(that are Hermitian adjoints of the step-up operators in (70)) in fact lower by two units the energy spectrum of $H_-(l)$. The quadratic ladder operators (70), (71) for the spectrum of $H_-(l)$ in fact coincide with those obtained recently by Dongpei [25] from factorisation considerations [34]. In fact Dongpei's *b* operators are the same as our *A* operators here (apart from an overall sign change) and his results [25] on the *l*-shift properties of these operators can be seen below to result from the Wigner system ladder relations (61*a*). In fact multiplying both sides of (61*a*) from the left by the projection operators $\frac{1}{2}(1 \pm \Sigma_3)$, one obtains

$$\tfrac{1}{2}(1 \pm \Sigma_3)[H(l+1), a^\pm(l+1)]_- = \pm\tfrac{1}{2}(1 \pm \Sigma_3)a^\pm(l+1).$$

Making use of (65) in the above and then eliminating the spinor parts leads to the following set of *l*-shift relations:

$$H_-(l)A^+(l+1) - A^+(l+1)H_-(l+1) = A^+(l+1) \tag{72a}$$

$$H_-(l+1)A^+(-(l+1)) - A^+(-(l+1))H_-(l) = A^+(-(l+1)) \tag{72b}$$

$$A^-(-(l+1))H_-(l+1) - H_-(l)A^-(-(l+1)) = A^-(-(l+1)) \tag{72c}$$

$$A^-(l+1)H_-(l) - H_-(l+1)A^-(l+1) = A^-(l+1). \tag{72d}$$

Turning now to the construction of the 1D SUSY isotonic system with unbroken SUSY, one just invokes analogy with (56) and (57) and effects the substitution in these of $\boldsymbol{\sigma} \cdot \boldsymbol{L} + 1 \to l+1$ (now any positive integer), $r \to x$ and $\partial/\partial r + 1/r \to \mathrm{d}/\mathrm{d}x$ so as to obtain

$$H_{\mathrm{SS}} = H(l+1) - \tfrac{1}{2}\Sigma_3[1 + 2(l+1)\Sigma_3] \tag{73a}$$

$$= \tfrac{1}{2}[a^-(l+1), a^+(l+1)]_+ - \tfrac{1}{2}\Sigma_3[a^-(l+1), a^+(l+1)]_- \tag{73b}$$

$$= [Q_-, Q_+]_+ \tag{73c}$$

$$= \begin{bmatrix} A^+(l+1)A^-(l+1) & 0 \\ 0 & A^-(l+1)A^+(l+1) \end{bmatrix} \tag{73d}$$

$$(Q_-)^2 = (Q^+)^2 = 0 \tag{73e}$$

with

$$Q_- = \tfrac{1}{2}(1 - \Sigma_3)a^-(l+1) = \Sigma_- A^-(l+1) \tag{74a}$$

$$Q_+ = (Q_-)^\dagger = \tfrac{1}{2}(1 + \Sigma_3)a^+(l+1) = \Sigma_+ A^+(l+1) \tag{74b}$$

$$Q_- \begin{bmatrix} \phi_{\mathrm{B}} \\ \phi_{\mathrm{F}} \end{bmatrix} = Q_+ \begin{bmatrix} \phi_{\mathrm{B}} \\ \phi_{\mathrm{F}} \end{bmatrix} = 0 \to \phi = \begin{bmatrix} \phi_{\mathrm{B}} \\ \phi_{\mathrm{F}} \end{bmatrix} \propto \begin{bmatrix} x^{l+1}\exp(-\frac{1}{2}x^2) \\ 0 \end{bmatrix}. \tag{74c}$$

In (74), $\Sigma_\mp$ are defined as in (8) and $A^\pm(l+1)$ as in (68).

Now consider d-dimensional ($d \neq 1$) radial oscillator Hamiltonians in the physical representation $R(r)$

$$H_-^R = \frac{1}{2}\left(\frac{-\mathrm{d}^2}{\mathrm{d}r^2} - \frac{d-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l_d(l_d+d-2)}{r^2} + r^2\right) \qquad H_-^R R_-(r) = E_- R_-(r). \tag{75}$$

The observation that $H_-(l)$ of (61b) with x replaced by r ($0<r<\infty$) coincides with the transformed forms [35] H_-^χ in the χ representation

$$\chi_-(r) = r^{(1/2)(d-1)} R_-(r) \qquad H_-^\chi = r^{(1/2)(d-1)} H_-^R r^{(1/2)(1-d)} \tag{76}$$

for angular momentum $l_d = l - \frac{1}{2}(d-3)$, $l_d = 0, 1, 2, \ldots$, of the radial oscillator leads to the analogous defining of the associated Wigner system Hamiltonian by $H^\chi(c/2) = H^\chi(l+1) = H^\chi(l_d + \frac{1}{2}(d-1))$, i.e.

$$H^\chi(l_d+\tfrac{1}{2}(d-1)) = \begin{bmatrix} H_-^\chi(l_d+\frac{1}{2}(d-3)) & 0 \\ 0 & H_+^\chi(l_d+\frac{1}{2}(d-3)) = H_-^\chi(l_d+\frac{1}{2}(d-1)) \end{bmatrix} \tag{77a}$$

$$= [a_\chi^-(l_d+\tfrac{1}{2}(d-1)), a_\chi^+(l_d+\tfrac{1}{2}(d-1))]_+ \tag{77b}$$

$$H_-^\chi(l_d+\tfrac{1}{2}(d-3)) = \frac{1}{2}\left(\frac{-\mathrm{d}^2}{\mathrm{d}r^2} + r^2 + \frac{1}{r^2}[l_d+\tfrac{1}{2}(d-3)][l_d+\tfrac{1}{2}(d-1)]\right) \tag{77c}$$

$$[H^\chi(l_d+\tfrac{1}{2}(d-1)), a_\chi^\mp(l_d+\tfrac{1}{2}(d-1))]_- = \pm a_\chi^\mp(l_d+\tfrac{1}{2}(d-1)). \tag{77d}$$

Though the χ-representation Wigner ladder operators $a_\chi^\mp(l_d+\frac{1}{2}(d-1))$ are not mutually adjoint and hence $H^\chi(l_d+\frac{1}{2}(d-1))$ is not Hermitian, the similarity transformed operators

$$a_R^\mp(c/2 = l_d+\tfrac{1}{2}(d-1)) = r^{(1/2)(1-d)} a_\chi^\pm(c/2 = l_d+\tfrac{1}{2}(d-1)) r^{(1/2)(d-1)} \tag{78a}$$

in the physical R-representation are indeed mutually adjoint:

$$a_R^\pm(c/2 = l_d+\tfrac{1}{2}(d-1)) = [a_R^\mp(c/2 = l_d+\tfrac{1}{2}(d-1))]^\dagger \tag{78b}$$

and hence the Wigner Hamiltonian in this physical representation

$$H^R(l_d+\tfrac{1}{2}(d-1)) = r^{(1/2)(1-d)} H^\chi(l_d+\tfrac{1}{2}(d-1)) r^{(1/2)(d-1)} \tag{79a}$$

$$= [a_R^-(l_d+\tfrac{1}{2}(d-1)), a_R^+(l_d+\tfrac{1}{2}(d-1))]_+ \tag{79b}$$

is Hermitian. Noting this fact and also that $c = 2(l_d+\frac{1}{2}(d-1)) > 0$ with $d \neq 1$ as is assumed here, the algebraic technique of section 2 may be applied immediately to the present case to obtain first the energy eigenfunctions $\chi_-^{(m)}(l_d+\frac{1}{2}(d-3))$ of $H_-^\chi(l_d+\frac{1}{2}(d-3))$ as given by (37) with the energy spectrum $E_-^{\chi(m)}(l_d+\frac{1}{2}(d-3))$ as given by (31c) with the due substitution of $c/2$ in these by $l_d+\frac{1}{2}(d-1)$. The physical representation spectral resolution follows the similarity transformation in (79a) to yield

$$R_-^{(m)}(l_d+\tfrac{1}{2}(d-3); r) = r^{(1/2)(1-d)} \chi_-^{(m)}(l_d+\tfrac{1}{2}(d-3); r) = R_{Nl_d}(r) \propto r^{l_d} \exp(-\tfrac{1}{2}r^2) L_{(1/2)(N-l_d)=m}^{(l_d+(1/2)d-1)}(r^2) \tag{80a}$$

$$E_-^{R(m)}(l_d+\tfrac{1}{2}(d-3)) = (E_-)_{Nl_d} = E^{(0)}(l_d+\tfrac{1}{2}(d-1)) + 2m = l_d+\tfrac{1}{2}(d-1)+\tfrac{1}{2}+2m$$

$$= l_d + \tfrac{1}{2}d + 2m \qquad m = 0, 1, 2, \ldots \tag{80b}$$

$$= N + \tfrac{1}{2}d \qquad N = l_d, l_d+2, \ldots \tag{80c}$$

where N is the principal quantum number.

The construction of the SUSY system from the Wigner system in the χ representation for the d-dimensional radial oscillator ($d \neq 1$) follows the same lines as in equations (73) and (74) for the isotonic case by effecting the substitution of $x \to r$ and $c/2 = l+1 \to l_d + \frac{1}{2}(d-1)$, but is not repeated here.

The usual one-dimensional case is realised from (11) for the special choice of $c/2 = 0$. For this unique case the generalised commutation relations (13) and (14) are sharpened to the usual canonical ones. Retaining the super-realisations (10) for the ladder operators with $c/2 = 0$, i.e.

$$a^{\pm}(0) = \frac{1}{\sqrt{2}}\left(\pm\Sigma_1 \frac{\mathrm{d}}{\mathrm{d}x} - \Sigma_1 x\right) = \frac{1}{\sqrt{2}}\Sigma_1\left(\pm\frac{\mathrm{d}}{\mathrm{d}x} - x\right) \tag{81}$$

the Wigner Hamiltonian defined in (11) for this case becomes

$$H(0) = \begin{bmatrix} H_-(-1) & 0 \\ 0 & H_+(-1) = H_-(0) \end{bmatrix} = \tfrac{1}{2}\left(\frac{-\mathrm{d}^2}{\mathrm{d}x^2} + x^2\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{82}$$

This form is different from those treated earlier with $c/2 > 0$ for which cases the bosonic and fermionic sector Hamiltonians could not be identical (see (11c) and (11d)) whereas for the present case, equation (82), the sector Hamiltonians indeed become identical to the usual one-dimensional Hamiltonians and hence possess identical energy spectra. Thus, every level (including the ground state) of $H(0)$ is doubly degenerate, the degeneracy being labelled by the both admissible values ± 1 of Σ_3-parity, i.e. by the eigenvalues $\pm$ of Σ_3. (Equivalently, one could label the degeneracy by the eigenvalues ± 1 of Σ_1 or Σ_2 which also commute with $H(0)$, but we adopt the Σ_3-parity convention.) Starting from the ground state of $H(0)$ with even Σ_3-parity (the other linearly independent ground state has odd Σ_3-parity), the repeated application of the operator $a^+(0)$ to this state will effect the construction of the excited states of $H(0)$ with alternating Σ_3-parity in view of (29). Hence, we obtain by applying the same algebraic method of section 2 to the present case of $c/2 = 0$, the following expressions for the Nth excited states with specific Σ_3-parities and the corresponding energy levels of $H(0)$:

$$\psi^{(N=2m)}(0) = \begin{bmatrix} \psi_-^{(N=2m)}(-1) \\ 0 \end{bmatrix} \propto \begin{bmatrix} \exp(-\frac{1}{2}x^2) L^{(-1/2)}_{m=(1/2)N}(x^2) \\ 0 \end{bmatrix}$$
$$\propto \begin{bmatrix} \exp(-\frac{1}{2}x^2) H_{N=2m}(x) \\ 0 \end{bmatrix} \tag{83a}$$

$$E^{(N=2m)}(0) = E_-^{(N=2m)}(-1) = \tfrac{1}{2} + 2m \qquad m = 0, 1, 2, \ldots \tag{83b}$$

$$\psi^{(N=2m+1)}(0) = \begin{bmatrix} 0 \\ \psi_+^{(N=2m+1)}(-1) \end{bmatrix} \propto \begin{bmatrix} 0 \\ x\exp(-\frac{1}{2}x^2) L^{(1/2)}_{m=(1/2)(N-1)}(x^2) \end{bmatrix}$$
$$\propto \begin{bmatrix} 0 \\ \exp(-\frac{1}{2}x^2) H_{N=2m+1}(x) \end{bmatrix} \tag{83c}$$

$$E^{(N=2m+1)}(0) = E_+^{(N=2m+1)}(-1) = \tfrac{1}{2} + 2m + 1 \qquad m = 0, 1, 2, \ldots. \tag{83d}$$

Noting that Σ_1 commutes with $H(0)$, the other set of states linearly independent to those in (83a) and (83c) with opposite Σ_3-parities can obviously be obtained by Σ_1 operation on the states in (83a) and (83c). Equations (83) verify with the well known results for the one-dimensional case, wherein however, as a by-product of our algebraic method, we have rederived the proportionalities (83a) and (83c) existing [31] between the Hermite and the generalised Laguerre polynomials involved in these equations.

6. Conclusions

In this paper we have pointed out the efficacy of a super-realised WH algebra as an effective operator tool for easier spectral resolution of general oscillator Hamiltonians like those of the 3D isotropic, 1D-isotonic and d-dimensional radial oscillator systems. In our method these Hamiltonians were incorporated in the bosonic sectors of the corresponding Wigner Hamiltonian and their complete energy spectrum and eigenfunctions obtained by the WH algebra inspired generalisation in section 2 of the ladder operator method for the usual one-dimensional oscillator. Such a direct algebraic method elucidated in this paper for these systems has not, in our opinion, been reported so far in the literature and we believe it to be a welcome addition to the existing SUSY-inspired operator method ([3-5]) for SUSY shape-invariant potentials of which the oscillator potentials treated here also form members [4]. The procedure detailed here proves highly profitable for simpler algebraic treatment, as we shall show in subsequent publications, of other quantum mechanical systems with underlying oscillator connections like for example those of a non-relativistic or relativistic electron in a Coulomb potential or of certain 3D SUSY oscillator models of the type of Celka and Hussin [20]. Also one of the authors (JJ) will demonstrate elsewhere how a significant percentage of other known SUSY shape-invariant potentials (see [4] for a list of such potentials) do possess hidden connections with a Wigner oscillator and hence are amenable to simpler algebraic treatment on the basis of the operator technique of this paper.

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References

[1] Witten E 1981 *Nucl. Phys.* B **185** 513-54
[2] Nicolai H 1976 *J. Phys. A: Math. Gen.* **9** 1497-506
[3] Gendenshteĭn L 1983 *JETP Lett.* **38** 356-9
[4] Dutt R, Khare A and Sukhatme U P 1988 *Am. J. Phys.* **56** 163-8
[5] Dutt R, Khare A and Sukhatme U P 1986 *Phys. Lett.* **181B** 295-8
[6] Dabrowska J W, Khare A and Sukhatme U P 1988 *J. Phys. A: Math. Gen.* **21** L195-200
[7] Cooper F, Ginocchio N and Khare A 1987 *Phys. Rev.* D **36** 2458-73
[8] Cooper F, Ginocchio J N and Wipf A 1988 *Phys. Lett.* **129A** 145-7
[9] Lévai G 1989 *J. Phys. A: Math. Gen.* **22** 689-702
[10] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** L57-61, L697-701
[11] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917-36, 2937-55
[12] Andrianov A A, Borisov N V, Eides M I and Ioffe M V 1985 *Phys. Lett.* **109A** 143-8
[13] Wigner E P 1950 *Phys. Rev.* **77** 711-2
[14] Yang L M 1951 *Phys. Rev.* **84** 788-90
[15] O'Raifeartaigh L and Ryan C 1963 *Proc. R. Irish Acad.* **62** 93-115

[16] Boulware D G and Deser S 1963 *Nuovo Cimento* **30** 230-4
[17] Ohnuki Y and Kamefuchi S 1978 *J. Math. Phys.* **19** 67-78
[18] Mukunda N, Sudarshan E C G, Sharma J K and Mehta C L 1980 *J. Math. Phys.* **21** 2386-94
[19] Watanabe S 1988 *Prog. Theor. Phys.* **80** 947-52
[20] Celka P and Hussin V 1987 *Mod. Phys. Lett.* **2A** 391-5
[21] Ui H 1984 *Prog. Theor. Phys.* **72** 813-20
[22] Balantekin A B 1985 *Ann. Phys., NY* **164** 277-87
[23] Ui H and Takeda G 1984 *Prog. Theor. Phys.* **72** 266-84
[24] Beckers J, Dehin D and Hussin V 1987 *J. Phys. A: Math. Gen.* **20** 1137-54
[25] Dongpei Z 1987 *J. Phys. A: Math. Gen.* **20** 4331-6
[26] Camiz P, Gerardi A, Marchioro C, Presutti E and Scacciatelli E 1971 *J. Math. Phys.* **12** 2040-3
[27] Weissman Y and Jortner J 1979 *Phys. Lett.* **70A** 177-9
[28] Nieto M M and Simmons L M Jr 1979 *Phys. Rev.* D **20** 1332-41
[29] Green H S 1953 *Phys. Rev.* **90** 270-3
[30] Sharma J K, Mehta C L and Sudarshan E C G 1978 *J. Math. Phys.* **19** 2089-93
[31] Gradshteyn I S and Ryzhik I M 1965 *Table of Integrals, Series, and Products* (4th edn translation edited by A Jefferey) (New York: Academic) pp 1037-9
[32] Mathews P M and Venkatesan K 1986 *A Text Book of Quantum Mechanics* (New Delhi: Tata McGraw-Hill) pp 335-7
[33] Biedenharn L C 1983 *Found. Phys.* **13** 13-34
[34] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21-68
[35] Kostelecki V A, Nieto M M and Truax D R 1985 *Phys. Rev.* D **32** 2627-33